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On Irrational Covariants of certain Binary Forms.

By E. Study.

In the following pages we intend to study the most important irrational covariants of binary cubics and quartics and of some other special binary forms.

The subject was entered upon by the great originator of the theory of invariants, Cayley, and it has been brought to perfection in some respects, through the application of refined methods, by Clebsch and others. There are, however, quite a number of details which make further investigations desirable, especially when we consider the intimate connection of the subject with some parts of the modern theory of functions. For reasons of this kind it was almost inevitable to the author to work the whole subject over again. Presenting, in a carefully chosen system of notation, the result of this rather laborious task to the mathematical world, I hope that it will be useful to those who have to deal with the numerous applications of the binary quantics of the lowest orders.

Instead of summarizing our new results in detail, we prefer to signalize the points in which our way of looking at the matter differs from the usual one.

As to *algebraical equations*, Cayley's principal point of view seems to have been to derive solutions of the equations of the lowest degrees from the theory of invariants. He wrote an equation $f = 0$, say of the third degree, homogeneously, and based the solution upon the identical relation among the covariants of the cubic form f. Thus the introduction of irrational covariants, representing the linear factors of f, was to him chiefly an intermediate step in the investigation of the solution of $f = 0$, that is to say, of the *vanishing points* of the quantic f.

While sharing the general views as to the beauty and importance of Cayley's discovery, we venture to profess a different opinion concerning the source of this importance. The idea of solving the equation $f = 0$ by means of the theory of invariants is, as we are inclined to think, open to some criticism,* even if we do not mind the fact that this solution had been known long before anybody thought of invariants. On the other hand, the linear factors of f are roots of a certain equation $F=0$, containing the variables of f as parameters in their coefficients. This equation is, of course, *not* furnished by the ordinary treatment of the equation $f=0$; and even in the theory of invariants it has not yet received much attention. It seems to us, however, to be the main feature of the theory; we shall therefore endeavor to point out distinctly its very special properties.

We have here a problem in which the theory of equations and the theory of invariants are superposed, or rather we have two different problems:

1st. To decide whether or not algebraic equations $F=0$ exist, the coefficients of which belong to the system of rational invariants and covariants of a given system of quantics $f_1, f_2, \ldots.$, and the roots of which represent divisors of a form f occurring in this system; and, if possible, to determine these equations, especially the "simplest" ones (according to properly chosen definitions of "simplicity").

2d. To solve these equations.

The reader will notice that the first question does not always admit of an affirmative answer, and that also several equally simple solutions may exist, differing by factors of proportionality, and not reducible to one another by rational substitutions.† In the actual treatment of special cases it will not be necessary to keep separated the two steps, or to urge the definition of the

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* Namely, the covariants do not seem to act any *necessary* part in the solution of a given equation; actually Cayley's form of the solution does not present itself readily in Galois' theory.

To show the difference of the standpoints in a simple example, we may compare Cayley's solution of a quadratic equation, which involves a variable parameter p (or rather a pair of homogeneous parameters $p_1 : p_2$) with the ordinary solution, which is free from this complication. The parameter is accounted for by the fact, that in the system of covariants of a binary quadratic form f no equation $F=0$ exists, the roots of which would be linear factors of f; whereas such an equation can be formed in the simultaneous system of the quadratic form f and a linear form p. The parameter would be utterly superfluous if merely the solution of the equation $f=0$ were in question.

† An interesting example is furnished by the combinant of the eighth order in the theory of a binary quartic. We are to decompose this form in no less than *four* different manners into a pair of biquadratic factors, each representing four equianharmonic points. The irrational covariants defining these factors have very different properties. The geometrical interpretation of a quantic by the mere group of its vanishing points fails entirely to give an account of such occurrences.

"simplest" solution of the first question; this definition is furnished implicitly by the investigation itself.

The said equations $F=0$ contain variable parameters in their coefficients. Now, in all cases dealt with in this paper they enjoy a remarkable property: *they have resolvents containing a smaller number of parameters.*

Since very little is known as to equations with parameters in general, we may claim some interest for our developments, however special they may appear, from the mere standpoint of the theory of equations.

Our notation is the so-called symbolical one, which is by far the most convenient notation at least for special researches of this character. The reader is supposed to be familiar with the methods applied in the standard work of Clebsch, "Theorie der binären Formen" (Leipzig, 1872), especially with the properties of the *rational* covariants of binary cubics and quartics contained in Chapter IV. Instead of Clebsch, also Gordan's "Vorlesungen über Invariantentheorie" (herausgegeben von Kerschensteiner, II, Leipzig, 1887) may be consulted.

It must be mentioned that we differ slightly from both authors in the notation of binary variables. We denote a binary quantic by $(ax)^n=(a_1x_2-a_2x_1)^n$, instead of writing $a_x^n=(a_1x_1+a_2x_2)^n$. Thus we avoid the introduction of the so-called principle of contragredience, which has no genuine right of existence in the binary domain.

I.—THE CUBIC.

In the theory of a binary cubic we introduce, for certain reasons which will appear later on, a system of notation slightly different from that used by Clebsch. Thus denoting the forms

$$f,\ \Delta,\ Q,\ R$$

of Clebsch by

$$p,\ 2\delta,\ q,\ 2r,$$

we have the following forms, which constitute what is called the complete system of p:

$$\begin{aligned} p &= (px)^3=(p'x)^3=\ldots\ldots,\\ \delta &= (\delta x)^2=\tfrac{1}{2}(pp')^2(px)(p'x),\\ q &= (qx)^3=2(p\delta)(px)^2(\delta x)=-(pp')^2(pp'')(p'x)(p''x)^2,\\ r &= 2(\delta\delta')^2=\tfrac{1}{2}(pq)^3. \end{aligned} \tag{1}$$

These forms are connected by the syzygy

$$rp^2 + 4\delta^3 + q^2 = 0, \tag{2}$$

viz.

$$-4\delta^3 = \{q + \sqrt{-r}.p\}\{q - \sqrt{-r}.p\}. \tag{3}$$

Consequently we are able to decompose $-\delta$ into its linear factors (σx) and (τx), defining these forms by means of the equations

$$\begin{gathered}(\sigma x) = \sqrt[3]{\frac{q + \sqrt{-r}.p}{2}}, \quad (\tau x) = \sqrt[3]{\frac{q - \sqrt{-r}.p}{2}}, \\ (\sigma x)(\tau x) = -\delta = -(\delta x)^2.\end{gathered} \tag{4}$$

These forms are *irrational covariants* of f; *they are roots of the sextic equation*

$$y^6 - qy^3 - \delta^3 = 0. \tag{5}$$

Now, from the equations

$$\begin{aligned}(\sigma x)^3 + (\tau x)^3 &= q, \\ (\sigma x)^3 - (\tau x)^3 &= \sqrt{-r}.f,\end{aligned} \quad (\sigma x)(\tau x) = -(\delta x)^2$$

we derive $(\sigma\tau)^3 = r\sqrt{-r}$, $(\sigma\tau)^2 = -r$, that is to say,

$$(\sigma\tau) = -\sqrt{-r}. \tag{6}$$

The linear forms (σx), (τx) thus defined evidently give rise to an unlimited number of irrational covariants of the cubic p contained in the form $\lambda(\sigma x)^a + \mu(\tau x)^a$. Among these forms we find certain linear forms proportional to linear factors of p, and others proportional to those of q, or, as we may say briefly, we find among our irrational covariants the linear factors of p and q.

Let $\sqrt{-3}$ be an arbitrary but definite value of the square root of -3, and denote by ε_1, ε_2, ε_3 the cubic roots of unity, derived from 1, $\varepsilon = \frac{-1+\sqrt{-3}}{2}$, $\varepsilon^2 = \frac{-1-\sqrt{-3}}{2}$ by any *cyclical* permutation; finally by $\bar{\varepsilon}_1$, $\bar{\varepsilon}_2$, $\bar{\varepsilon}_3$ their conjugate values 1, ε^2, ε, arranged in the corresponding order. Then the equations

$$\begin{aligned}(\sigma\tau).(\lambda x) &= \bar{\varepsilon}_1.(\sigma x) - \varepsilon_1.(\tau x), \\ (\sigma\tau).(\mu x) &= \bar{\varepsilon}_2.(\sigma x) - \varepsilon_2.(\tau x), \\ (\sigma\tau).(\nu x) &= \bar{\varepsilon}_3.(\sigma x) - \varepsilon_3.(\tau x),\end{aligned} \tag{7}$$

will define certain irrational covariants of the degree -1, fulfilling the conditions

$$0 = (\lambda x) + (\mu x) + (\nu x), \tag{8}$$

$$\begin{aligned} 3\delta &= r\{(\mu x)(\nu x) + (\nu x)(\lambda x) + (\lambda x)(\mu x)\} \\ &= -\frac{r}{2}\cdot\{(\lambda x)^2 + (\mu x)^2 + (\nu x)^2\}, \end{aligned} \tag{9}$$

$$p = r.(\lambda x)(\mu x)(\nu x) = \frac{r}{3}.\Sigma(\lambda x)^3, \tag{10}$$

$$3\sqrt{-3}.q = r\cdot\sqrt{-r}.\{(\mu x)-(\nu x)\}\{(\nu x)-(\lambda x)\}\{(\lambda x)-(\mu x)\}, \tag{11}$$

$$-\frac{\sqrt{-3}}{(\sigma\tau)} = \frac{\sqrt{-3}}{\sqrt{-r}} = (\mu\nu) = (\nu\lambda) = (\lambda\mu). \tag{12}$$

Hence the forms (λx), (μx), (νx) *are the roots of the cubic equation*

$$rx^3 - * + 3\delta.x - p = 0.^* \tag{13}$$

Their immediate expressions by means of radicals are given by the formulæ

$$\begin{gathered} x = \sqrt[3]{\frac{rp + q\sqrt{-r}}{2r^2}} + \sqrt[3]{\frac{rp - q\sqrt{-r}}{2r^2}}, \\ \sqrt[3]{(+)}\cdot\sqrt[3]{(-)} = -r.\delta. \end{gathered} \tag{14}$$

In the same way the decomposition of the cubic covariant q is connected with three irrational covariants $(\lambda'x)$, $(\mu'x)$, $(\nu'x)$ of the degree $+1$.

We may first introduce the notation

$$(\sigma'x) = \frac{(\sigma x)}{(\sigma\tau)}, \quad (\tau'x) = \frac{(\tau x)}{(\tau\sigma)}. \tag{4b}$$

These forms are roots of the sextic equation

$$r^3.y'^6 - r^2p.y'^3 - \delta^3 = 0. \tag{5b}$$

Further, we define three linear forms $(\lambda'x)$, $(\mu'x)$, $(\nu'x)$ by means of the formulæ

$$(\sigma'\tau').(\lambda'x) = \bar{\varepsilon}_1(\sigma'x) - \varepsilon_1(\tau'x), \text{ etc.} \tag{7b}$$

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[*] This equation is evidently the *simplest* one among all those by which linear forms, proportional to the linear factors of p, can be defined. It has the lowest order possible, namely, *three*, and the degrees of the coefficients do not surpass four. By these properties it is defined, if we do not mind substitutions of the form $x = \rho^{x'}$, where ρ is a rational number. But it is even the simplest equation with respect to the numerical coefficients.

The linear factors of p used by Cayley and Clebsch are roots of an equation of the *sixth* degree. Similar remarks hold in other cases.

or
$$-(\lambda' x) = \breve{\varepsilon}_1 (\sigma x) + \varepsilon_1 (\tau x), \text{ etc.,} \tag{7c}$$
or finally,
$$-\sqrt{-3}\,.(\lambda' x) = (\sigma\tau).\{(\mu x) - (\nu x)\}, \tag{7d}$$

etc., with cyclical permutation of λ, μ, ν.

Then we have

$$0 = (\lambda' x) + (\mu' x) + (\nu' x), \tag{8b}$$

$$\begin{aligned} 3\delta &= (\mu' x)(\nu' x) + (\nu' x)(\lambda' x) + (\lambda' x)(\mu' x) \\ &= -\tfrac{1}{2}\{(\lambda' x)^2 + (\mu' x)^2 + (\nu' x)^2\}, \end{aligned} \tag{9b}$$

$$q = (\lambda' x)(\mu' x)(\nu' x) = \tfrac{1}{3}\Sigma\,(\lambda' x)^3, \tag{10b}$$

$$3\sqrt{-3}\,.p = \frac{1}{\sqrt{-r}}\{(\mu' x) - (\nu' x)\}\{(\nu' x) - (\lambda' x)\}\{(\lambda' x) - (\mu' x)\} \tag{11b}$$

$$-\frac{\sqrt{-3}}{(\sigma'\tau')} = -\sqrt{-3}\sqrt{-r} = (\mu'\nu') = (\nu'\lambda') = (\lambda'\mu'). \tag{12b}$$

Hence the forms $(\lambda' x)$, $(\mu' x)$, $(\nu' x)$ *are the roots of the cubic equation*

$$z^3 - * + 3\delta.z - q = 0. \tag{13b}$$

Their expressions in terms of radicals are given by the formulæ

$$\begin{aligned} z &= \sqrt[3]{\frac{q - p\sqrt{-r}}{2}} + \sqrt[3]{\frac{q + p\sqrt{-r}}{2}}, \\ &\sqrt[3]{(-)}.\sqrt[3]{(+)} = -\delta. \end{aligned} \tag{14b}$$

These formulæ put the reciprocity between the forms p and q into evidence: *Our formulæ are interchanged among each other when we replace*

$$p\,,\quad \delta\,,\qquad q\,,\quad r\,,\qquad (\sigma x),\ (\sigma' x),\ (\lambda x),\qquad (\lambda' x),\ \textit{etc.},$$

respectively by

$$\frac{q}{r}\,,\ \frac{\delta}{r}\,,\ -\frac{p}{r}\,,\ \frac{1}{r}\,,\ -(\sigma' x),\ (\sigma x),\ (\lambda' x),\ -(\lambda x)\,,\ \textit{etc.}^*$$

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[*] $\sqrt{-r}$ has to be replaced by $-\frac{1}{\sqrt{-r}}\cdot-$. It is this law of reciprocity by which the previously mentioned change of notation is suggested. We should not have obtained so simple numerical coefficients in our formulæ if we had operated with the forms Δ and R of Clebsch instead of our δ and r. It is worth noticing that the very same change of notation presents itself still from a quite different point of view. We have avoided the common divisor 2 which appears in the expressions of Δ and R in terms of coefficients of the cubic. (See Clebsch, Binäre Formen, p. 114.)

We recognize in this law of reciprocity the expression of a linear transformation of the binary domain, transforming the vanishing points of p into those of q and *vice versa.*

The equations (5), (5b), (13), (13b) contain a variable parameter $(x_1 : x_2)$ in their coefficients. But they are very special equations of this character. The root is in all cases an *integral* function of the parameter $x_1 : x_2$; in other terms, the Riemann surfaces belonging to the algebraic functions defined by these equations break up into separated leaves. Secondly, these equations have a quadratic resolvent $(\lambda^2 + r = 0)$ which is entirely free from the parameter $(x_1 : x_2)$.

The irrational covariants defined by (5) and (5b) represent in either case the linear factors of δ; but whereas by the solution of (5) δ itself is decomposed into two linear factors, the other equation (5b) decomposes the product $r.\delta$.

In the same way by the solution of (13) not p itself is decomposed into three linear factors, but the product r^2p. The decomposition of p itself into a product of three irrational linear covariants would evidently depend upon the solution of an equation of the *ninth* degree, since the new radical $\sqrt[3]{r}$ has to be introduced. But if the cubic is considered as the cubic covariant of another cubic, which is supposed to be rationally known, then it can be decomposed by the solution of a cubic equation, as is shown by the formula (13b). If we should extend our considerations to the theory of a binary *quintic*, we should reach, in the case of the so-called *canonizant*, another simplification. The vanishing points of this cubic can be represented by irrational covariants of the quintic, which depend upon the roots of a cubic equation whose coefficients are mere invariants (viz. free from the parameter $(x_1 : x_2)$).

Our formulæ contain, of course, also the decomposition of the *sextic* $s^2.rp^2 + t^2.q^2$, wherein $s^2 : t^2$ denotes an arbitrary ratio, into its linear factors.

We have

$$s^2.rp^2 + t^2.q^2 = (t.q + s.\sqrt{-r}.p)(t.q - s.\sqrt{-r}.p). \tag{15}$$

Hence, denoting the two factors of this expression by p_1 and p_2, we have

$$\left.\begin{aligned} p_1 &= t.q + s.\sqrt{-r}.p, \\ \delta_1 &= (t^2 - s^2).r.\delta, \\ q_1 &= (t^2 - s^2).r\sqrt{-r}.(s.q + t\sqrt{-r}.p), \\ r_1 &= (t^2 - s^2)^2.r^3. \end{aligned}\right\} \tag{16}$$

The last equation enables us to define

$$\sqrt{-r_1} = (t^2 - s^2).r\sqrt{-r}, \tag{16b}$$

whence

$$\begin{aligned}(\sigma_1 x) &= \varepsilon_\kappa \sqrt[3]{(s^2 - t^2)(s + t)}.\sqrt{-r}.(\sigma x), \\ (\tau_1 x) &= \bar{\varepsilon}_\kappa \sqrt[3]{(s^2 - t^2)(s - t)}.\sqrt{-r}.(\tau x), \\ (\sigma_1 x).(\tau_1 x) &= (t^2 - s^2).r.(\sigma x).(\tau x).\end{aligned} \tag{17}$$

From these equations we derive the expressions of $(\lambda_1 x)$, etc. Replacing $(\lambda_1 x)$ by $(\bar{\lambda}_1 x) = (t^2 - s^2).r.(\lambda_1 x)$, and denoting the two cube roots in (17) by R and $\bar{R}$, we find

$$(\lambda_1 x) = \tfrac{1}{3}\sqrt{-r}.\left\{(\varepsilon_1 R + \bar{\varepsilon}_1 \bar{R})(\lambda x) + (\varepsilon_2 R + \bar{\varepsilon}_2 \bar{R})(\mu x) + (\varepsilon_3 R + \bar{\varepsilon}_3 \bar{R})(\nu x)\right\}, \tag{18}$$

etc.; $(\bar{\mu}_1 x)$ and $(\bar{\nu}_1 x)$ are obtained by means of a cyclical permutation of $\varepsilon_1, \varepsilon_2, \varepsilon_3$, whereas $(\bar{\lambda}_2 x)$ is obtained by interchange of ε_κ and $\bar{\varepsilon}_\kappa$.

The quantities $(\bar{\lambda}_1 x)$, $(\bar{\lambda}_2 x)$, *etc., are the roots of the sextic equation*

$$\{\bar{x}^3 + 3\,(t^2 - s^2)\,\delta.\bar{x} - (t^2 - s^2).t.q\}^2 + (t^2 - s^2)^2.s^2.r.p^2 = 0. \tag{19}$$

It is a remarkable property of this equation that when the roots of (13) are known, its solution is effected by extraction of radicals *depending merely upon the parameters* s, t. Solving it directly by means of Cardano's formula, we find

$$\begin{aligned}\bar{x} &= \sqrt[3]{(t^2 - s^2)(t + s)}.\sqrt[3]{\frac{q \pm \sqrt{-r}.p}{2}} \\ &+ \sqrt[3]{(t^2 - s^2)(t - s)}.\sqrt[3]{\frac{q \mp \sqrt{-r}.p}{2}}\end{aligned} \tag{20}$$

This is easily seen to be in accordance with (17) and (18), the roots having the values $-R$, $\bar{R}$, (σx), (τx) respectively.

Another remarkable property is shown by the formula (18), namely, the sum of the coefficients of (λx), (μx), (νx), in (18) is zero; and the different roots of (19) are obtained when we exchange these coefficients in all manners possible. We do not insist upon the evident geometrical interpretation of this fact; but we find worth noticing the special form our result assumes when we write the parameters occurring in (15) in a peculiar manner.

Denote a set of three quantities, the sum of which is zero, by e_1, e_2, e_3, and write, as is done in Weierstrass's theory of elliptic functions,

$$\begin{gathered} e_2 e_3 + e_3 e_1 + e_1 e_2 = -\tfrac{1}{2}(e_1^2 + e_2^2 + e_3^2) = -\tfrac{1}{4} g_2, \\ e_1 e_2 e_3 = \tfrac{1}{4} g_3, \\ (e_2 - e_3)(e_3 - e_1)(e_1 - e_2) = \tfrac{1}{4}\sqrt{g_2^3 - 27 g_3^2} = \sqrt{G}. \end{gathered}$$

Supposing now

$$e_\lambda = \frac{\sqrt{-3}}{6} . (\varepsilon_\lambda R + \bar{\varepsilon}_\lambda \bar{R}),$$

that is to say,

$$(\bar{\lambda}_1 x) = \frac{2\sqrt{-r}}{\sqrt{-3}} . \{e_1(\lambda x) + e_2(\mu x) + e_3(\nu x)\}, \text{ etc.,} \tag{18b}$$

we have

$$t^2 - s^2 = g_2, \ t = 4\frac{\sqrt{G}}{g_2}, \ s = -3\sqrt{-3} . \frac{g_3}{g_2},$$

and the equation (19) is transformed into

$$\{\bar{x}^3 + 3g_2 . \delta . \bar{x} - 4 . \sqrt{G} . q\}^2 - 27g_3^2 . r . p^2 = 0. \tag{19b}$$

By its solution the sextic

$$\begin{aligned} 16G.q^2 \ - \ 27g_3^2.rp^2 &= \\ = -g_2^3.rp^2 - \ 64G.\delta^3 \ &= \\ = \quad g_2^3.q^2 \ + 108g_3^2.\delta^3 & \end{aligned} \tag{16b}$$

is decomposed into its linear factors. The square root $\sqrt{G}$ can of course be avoided by introducing $\sqrt{G}.\bar{x}$ instead of $\bar{x}$.

II.—THE QUARTIC AND THE OCTAHEDRON.

We shall now make an investigation of certain irrational covariants of a binary quartic f and its sextic covariant t, the so-called octahedron, which we may consider also, as is well known, as an independent sextic F, satisfying the condition $(F, F)_4 = 0$.*

Before doing so, we put together the forms of the complete system of f, and certain relations among these quantics, to be used in our calculations. All these relations can be found among, or easily be derived from, the formulæ developed in the above-mentioned treatise of Clebsch (§40–51, §111), to which we refer for the proofs. As far as the theory of the sextic t is concerned, the reader may consult also Klein's "Icosahedron" (Leipzig, 1884, I, §5, 10, 12) and the literature quoted there.

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* Gordan-Kerschensterner, Binäre Formen, §19. Here and further on we use Gordan's abbreviation $(\psi, \chi)_\kappa$ for the bilinear covariant $(ab)^\kappa (ax)^{n-\kappa} (bx)^{m-\kappa}$ of two binary forms $\psi = (ax)^n$ and $\chi (bx)^m$.

§1. *Rational Covariants.*

Dealing with the rational covariants of a binary quartic f, we shall use a system of notation which has already been employed by Brioschi and H. Weber, and which is, for more than one reason, more convenient than the usual system of notation as employed by Clebsch and most of his followers. Thus denoting the forms

$$f,\ H,\ T,\ i\ ,\ j$$

of Clebsch respectively by

$$f,\ 2h,\ t,\ 2g_2,\ 6g_3,$$

we have the following forms, constituting the complete system of rational covariants of f:

$$\begin{aligned} f &= (ax)^4 = (a'x)^4 = \dots\dots = (a_1x_2 - a_2x_1)^4 = \dots\dots, \\ h &= (hx)^4 = (h'x)^4 = \dots\dots = \tfrac{1}{2}(aa')^2(ax)^2(a'x)^2 \quad, \\ t &= (tx)^6 = (t'x)^6 = \dots\dots = 2(ah)(ax)^3(hx)^3 \quad, \\ g_2 &= \tfrac{1}{2}(aa')^4, \ g_3 = \tfrac{1}{3}(ah)^4 = \tfrac{1}{6}(aa')^2(aa'')^2(a'a'')^2 \quad . \end{aligned} \tag{1}$$

These forms are connected by a single syzygy, namely,

$$t^2 + 4h^3 - g_2hf^2 + g_3f^3 = 0. \tag{2}$$

This most important formula is the clue to the whole theory of irrational covariants.

By the side of the fundamental invariants and covariants we are to consider certain combinations of the above forms. First, of course, the discriminant, which is at the same time an invariant (quasi-discriminant) of t:

$$G = \frac{g_2^3 - 27g_3^2}{16} = \frac{3(t,\ t)_6}{16}. \tag{3}$$

Further, we introduce a special sign for a certain covariant of the fourth order and fourth degree:

$$\phi = (\phi x)^4 = \tfrac{1}{2}(2g_2.h - 3g_3.f). \tag{4}$$

It has, among others, these important properties: it is the one form of the pencil $\varkappa f + \lambda h$ that is conjugate to f:

$$(f,\ \phi)_4 = (a\phi)^4 = 0. \tag{5}$$

It is the one form of the pencil $\varkappa f + \lambda h$ whose Hessian is proportional to the Hessian of f:

$$\tfrac{1}{2}(\phi,\ \phi)_2 = -\tfrac{4}{3}G.h. \tag{6}$$

It is a bilinear covariant of f and t:

$$(t, f)_3 = (ta)^3(tx)^3(ax) = (\phi x)^4. \tag{7}$$

It is connected with f, h and t by the identity

$$(t, h)_1 = \tfrac{1}{3} f.\phi. \tag{8}$$

There is, finally, a perfect reciprocity between f and $\dfrac{\sqrt{-3}}{2\sqrt{G}}.\phi$.

Let us consider now the forms of the complete system of the sextic t, which may, when treated as an independent form satisfying the relation $(t, t)_4 = 0$, also be denoted by the sign T.

The forms in question are

$$\begin{aligned} T &= t = (tx)^6, \\ \Phi &= \tfrac{1}{2}(T, T)_2 = \tfrac{1}{2}(tt')^2(tx)^4(t'x)^4, \\ \Psi &= 2(T, \Phi)_1 = -(tt')^2(tt'')(tx)^3(t'x)^4(t''x)^5, \\ R &= \tfrac{1}{36}(T, T)_6 = \tfrac{1}{36}(tt')^6 = \tfrac{4}{27}G. \end{aligned} \tag{9}$$

They, too, are connected by a syzygy:

$$RT^4 + 4\Phi^3 + \Psi^2 = 0.^* \tag{10}$$

The expressions of Φ and Ψ in terms of f and h are

$$\Phi = -\frac{g_2}{3}.h^2 + g_3.hf - \frac{g_2^2}{36}.f^2. \tag{11}$$

$$\Psi = 2g_3.h^3 - \frac{g_2^2}{3}.h^2f + \frac{g_2g_3}{2}.hf^2 + \left(\frac{g_2^3}{108} - \frac{g_3^2}{2}\right)f^2. \tag{12}$$

To these identities we may add the formula

$$9g_2.\Phi + 3\phi^2 + 4G.f^2 = 0, \tag{13}$$

containing another important property of the form ϕ.

§2. *The Irrational Invariants* e_λ, e_μ, e_ν.

Denoting by $t_{\kappa\lambda}$ and $G_{\kappa\lambda}$ the covariant t and the invariant G, derived from the form $\kappa f + \lambda h$ instead of f, and defining a homogeneous cubic function $\Omega(\kappa, \lambda)$ by the formula

$$4\Omega = 4\kappa^3 - g_2\kappa\lambda^2 - g_3\lambda^3, \tag{14}$$

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*See Clebsch, Binäre Formen, §111. Our considerations in §5 contain also a proof of this important identity.

we find, following Cayley and Clebsch, that

$$t_{\kappa\lambda} = \Omega . t, \quad G_{\kappa\lambda} = \Omega^2 . G. \tag{15}$$

This shows first that t and G are so-called *combinants* of the pencil $\kappa f + \lambda h$; second, that this pencil contains three forms of vanishing discriminant, which are, on account of the identical vanishing of $t_{\kappa\lambda}$, moreover, perfect squares.

These forms are given by the expressions

$$h + e_\lambda f, \quad h + e_\mu f, \quad h + e_\nu f,$$

if we understand e_λ, e_μ, e_ν to be the roots of the cubic equation

$$0 = 4e^3 - g_2 e - g_3. \tag{16}$$

The cubic resolvent of this equation, the roots of which are the differences $e_\mu - e_\nu$, $e_\nu - e_\lambda$, $e_\lambda - e_\mu$, is

$$0 = e'^3 + \tfrac{3}{2} g_2 . e' - \sqrt{G}. \tag{17}$$

The sign of the quantity $\sqrt{G}$ may, once for all, be explained by the formula

$$\sqrt{G} = (e_\mu - e_\nu)(e_\nu - e_\lambda)(e_\lambda - e_\mu). \tag{18}$$

e_λ, e_μ, e_ν are irrational covariants of f of the degree one, but they are not combinants of the pencil $\kappa f + \lambda h$.

Comparing (2) with (16), we notice (with Cayley) that

$$-t^2 = 4(h + e_\lambda f)(h + e_\mu f)(h + e_\nu f). \tag{19}$$

Hence we may decompose $\frac{t}{2}$ into a product of three quadratic forms l', m', n', denoting preliminarily by l' or $(l'x)^2$ the square root

$$\sqrt{-h - e_\lambda f}, \text{ (etc.)}.$$

These forms l', m', n' are irrational covariants of f, but they are not covariants of t; that is to say, they are not combinants of the pencil $\kappa f + \lambda h$. *There are, however, certain quadratic forms l, m, n, proportional to l', m', n', which enjoy these important properties.*

In order to find them, we put $l' = r_\lambda . l$, $m' = r_\mu . m$, $n' = r_\nu . n$, and try to determine the multipliers r so as to make the invariants of the forms l, m, n *numerical* and equal to each other. The forms thus obtained will be combinants.

We have evidently

$$(m'^2,\ n'^2)_1 = m'.n'.(m',\ n')_1,$$

and

$$(m'^2,\ n'^2)_1 = \tfrac{1}{2}(e_\mu - e_\nu).t = (e_\mu - e_\nu).l'm'n'.$$

$$\therefore\ r_\mu r_\nu.(m,\ n)_1 = r_\lambda.(e_\mu - e_\nu).l.$$

Hence, choosing

$$r_\lambda = -\sqrt{e_\nu - e_\lambda}\sqrt{e_\lambda - e_\mu},\ \ r_\mu = -\sqrt{e_\lambda - e_\mu}\sqrt{e_\mu - e_\nu},\ \ r_\nu = -\sqrt{e_\mu - e_\nu}\sqrt{e_\lambda - e_\mu},$$

we obtain a system of three forms l, m, n, which are covariants of t as well as of f. Their degree, in both cases, is *zero*.

These forms are the true central point in the theory of the most important irrational covariants of f and t; we therefore enter upon a more careful study of them.

§3. *The Quadratic Forms l, m, n.*

The quadratic forms $(lx)^2$, $(mx)^2$, $(nx)^2$ found in §2 fulfil, as a simple calculation will show, the following system of conditions:

$$\begin{aligned} (m,\ n)_1 &= (mn)(mx)(nx) = -(lx)^2, \\ (n,\ l)_1 &= (nl)(nx)(lx) = -(mx)^2, \\ (l,\ m)_1 &= (lm)(lx)(mx) = -(nx)^2, \end{aligned} \tag{20}$$

$$\begin{aligned} \tfrac{1}{2}(ll')^2 = \tfrac{1}{2}(mm')^2 = \tfrac{1}{2}(nn')^2 &= 1, \\ \tfrac{1}{2}(mn)(nl)(lm) &= 1, \\ (mn)^2 = (nl)^2 = (lm)^2 &= 0, \end{aligned} \tag{21}$$

from which we may derive the further formulæ

$$l^2 + m^2 + n^2 = 0, \tag{22}$$

$$\begin{aligned} (m^2,\ n^2)_1 &= -mnl, & (m^2,\ n^2) &= \tfrac{2}{3}l^2, \\ (m^2,\ n^2)_3 &= 0\quad, & (m^2,\ n^2)_4 &= -\tfrac{4}{3}, \\ (l^2,\ l^2)_2 &= \tfrac{2}{3}l^2\quad, & (l^2,\ l^2)_4 &= \tfrac{8}{3},\ \text{etc.} \end{aligned} \tag{23}$$

By the side of these forms l, m, n containing the variables x_1, x_2, we may consider their polars $(lx)(ly)$, etc., containing two sets of variables x, y. First we notice that

$$(lx)^2.(ly)^2 = (lx)(ly).(lx)(ly) + (xy)^2. \tag{24}$$

Further we may establish the following theorem, which contains a very important property of the forms l, m, n:

Denoting by s, x, y, z *any four points (sets of binary variables), the quadrilinear forms*

$$\begin{aligned} \mathfrak{X}_0 &= (sx).(yz) \quad , \\ \mathfrak{X}_1 &= (ls)(lx).(ly)(lz) \quad , \\ \mathfrak{X}_2 &= (ms)(mx).(my)(mz), \\ \mathfrak{X}_3 &= (ns)(nx).(ny)(nz) \quad , \end{aligned} \tag{25}$$

and the corresponding forms $\mathfrak{Y}_\kappa$, $\mathfrak{Z}_\kappa$, *derived from* $\mathfrak{X}_\kappa$ *by cyclical permutation of* x, y, z, *are connected by the linear equations*

$$0 = \begin{array}{llll} \mathfrak{X}_0 + \mathfrak{Y}_0 + \mathfrak{Z}_0, & \mathfrak{X}_1 + \mathfrak{Y}_2 + \mathfrak{Z}_3, & \mathfrak{X}_2 + \mathfrak{Y}_3 + \mathfrak{Z}_1, & \mathfrak{X}_3 + \mathfrak{Y}_1 + \mathfrak{Z}_2, \\ \mathfrak{X}_1 + \mathfrak{Y}_3 + \mathfrak{Z}_2, & \mathfrak{X}_0 - \mathfrak{Y}_1 + \mathfrak{Z}_1, & \mathfrak{X}_3 + \mathfrak{Y}_0 - \mathfrak{Z}_3, & -\mathfrak{X}_2 + \mathfrak{Y}_2 + \mathfrak{Z}_0, \\ \mathfrak{X}_2 + \mathfrak{Y}_1 + \mathfrak{Z}_3, & -\mathfrak{X}_3 + \mathfrak{Y}_3 + \mathfrak{Z}_0, & \mathfrak{X}_0 - \mathfrak{Y}_2 + \mathfrak{Z}_2, & \mathfrak{X}_1 + \mathfrak{Y}_0 - \mathfrak{Z}_1, \\ \mathfrak{X}_3 + \mathfrak{Y}_2 + \mathfrak{Z}_1, & \mathfrak{X}_2 + \mathfrak{Y}_0 - \mathfrak{Z}_2, & -\mathfrak{X}_1 + \mathfrak{Y}_1 + \mathfrak{Z}_0, & \mathfrak{X}_0 - \mathfrak{Y}_3 + \mathfrak{Z}_3. \end{array} \tag{26}$$

The first of these is the elementary identity

$$(sx)(yz) + (sy)(zx) + (sz)(xy) = 0\,;$$

at the others we arrive by properly chosen processes of polarization with respect to the quadratic forms l, m, n.*

Another no less important property of the forms l, m, n is:

The products $(mx)(my).(nx)(ny)$, $(nx)(ny).(lx)(ly)$, $(lx)(ly).(mx)(my)$ *as well as the product* $(lx)(ly).(mx)(my).(nx)(ny)$ *are polars*, that is to say, the application of the operator $\dfrac{\partial^2}{\partial x_1 \partial y_2} - \dfrac{\partial^2}{\partial x_2 \partial y_1}$ to these forms produces zero.

Defining the "composition" or "symbolical multiplication" of two bilinear forms $s = (\alpha x)(\beta y)$ and $s' = (\alpha' x)(\beta' y)$ by the formula $ss' = (\alpha x)(\alpha' \beta)(\beta' y)$ (preferable, in some respects, to the definition $ss' = (\alpha x)(\beta \alpha')(\beta' y)$), the manifoldness of all binary bilinear forms constitutes what is termed a system of complex numbers. Choosing as fundamental units the forms

$$\begin{aligned} \iota_0 &= (xy) \quad\quad , & \iota_1 &= (lx)(ly) \;, \\ \iota_2 &= (mx)(my), & \iota_3 &= (nx)(ny), \end{aligned}$$

we obtain a well-known system of formulæ: *The law of composition of our bilinear forms is identical with the multiplication rule of quaternions.*

The equations $\iota_\kappa = 0$ represent a group of four commutative linear transformations of the binary domain, which interchange, as we shall see in §12 more in detail, the vanishing points of each form of the pencil $\varkappa f + \lambda h$ among each other.*

---

* Compare the author's paper, Am. Journal, Vol. XVI (p. 156), No. 12–14.

This explains the symmetry among the four bilinear forms (xy), $(lx)(ly)$, $(mx)(my)$, $(nx)(ny)$, which appears in the first theorem of this section, and which we shall meet again in some other theorems.

§4. *Expressions of the Forms f, h, ϕ, t in Terms of l, m, n.*

The forms l^2, m^2, n^2 belong, according to their definition, to the pencil $\varkappa f + \lambda h$. We find them connected with the forms f and h by the identities:

$$\begin{aligned} l^2 &= \frac{h + e_\lambda f}{(e_\mu - e_\lambda)(e_\nu - e_\lambda)} = \\ &= -\frac{(e_\mu - e_\nu)h + e_\lambda(e_\mu - e_\nu)f}{(e_\mu - e_\nu)(e_\nu - e_\lambda)(e_\lambda - e_\mu)} = \frac{1}{\sqrt{G}} \cdot \left\{ (e_\mu^2 - e_\nu^2)f - (e_\mu - e_\nu)h \right\}, \end{aligned} \tag{27}$$

etc., or, in a different arrangement of the formulæ, and with a slight generalization,

$$(lx)^2.(ly)^2 + (mx)^2.(my)^2 + (nx)^2.(ny)^2 = 2(xy)^2, \tag{28}$$

$$\begin{aligned} e_\lambda.(lx)^2.(ly)^2 + e_\mu.(mx)^2.(my)^2 + e_\nu.(nx)^2.(ny)^2 &= (ax)^2(ay)^2, \\ e_\lambda^2.(lx)^2.(ly)^2 + e_\mu^2.(mx)^2.(my)^2 + e_\nu^2.(nx)^2.(ny)^2 &= \\ = (ax)^2 \frac{(aa')^2}{2} (a'y)^2 &= (hx)^2(hy)^2 + \tfrac{1}{3} g_2.(xy)^2. \end{aligned} \tag{29}$$

To these formulæ we add the analogous expressions:

$$\begin{aligned} \sum (e_\mu - e_\nu).(lx)^2.(ly)^2 &= -\frac{3}{2\sqrt{G}} \cdot (\phi x)^2(\phi y)^2, \\ \sum (e_\mu - e_\nu)^2.(lx)^2.(ly)^2 &= \frac{9}{4G} \cdot (\phi x)^2 \frac{(\phi\phi')^2}{2} (\phi y)^2 = \\ &= -3\{(hx)^2(hy)^2 - \tfrac{1}{3} g_2.(xy)^2\}. \end{aligned} \tag{30}$$

Instead of the expression of t, too, we may write down at once the expressions of some of its polars, to be used in our calculations,

---

* Compare Cayley, Math. Annalen, t. XV, p. 238; Stephanos, ib. Vol. XXII, p. 299, and the author's paper, Berichte der Königl. Sächs. Gesellschaft der Wissenschaften, 1889, p. 177, §10.

$$
\begin{aligned}
&(lx)(ly)\,.\,(mx)(my)\,.\,(nx)(ny) = \\
&= -\frac{[(mx)^2.(ny)^2]^2 - [(my)^2.(nx)^2]^2}{4\,(xy)} = \text{etc.} = -\frac{(tx)^3(ty)^3}{2\sqrt{G}},
\end{aligned} \tag{31}
$$

$$
\frac{(lx)^2.(mx)^2.(nx)^2}{3}\,.\left\{\frac{(ly)^2}{(lx)^2} + \frac{(my)^2}{(mx)^2} + \frac{(ny)^2}{(nx)^2}\right\} = -\frac{(tx)^4(ty)^2}{2\sqrt{G}}\,. \tag{32}
$$

Combining (31) with the well known identity

$$
2\,(xy)\,.\,(tx)^3(ty)^3 = (ax)^4.(hy)^4 - (ay)^4.(hx)^4, \tag{33}
$$

we have finally

$$
(xy)\,.\,(lx)(ly)\,.\,(mx)(my)\,.\,(nx)(ny) = \frac{(ax)^4(hy)^4 - (ay)^4(hx)^4}{-4\sqrt{G}}\,. \tag{34}
$$

The invariants e_λ, e_μ, e_ν being known, the forms l^2, m^2, n^2 are perfectly determined by the formulæ (27), whereas the forms l, m, n themselves are certain square roots of l^2, m^2, n^2.

There are, *under the said condition*, altogether *four* systems of values of l, m, n satisfying the relations established in §3, our formulæ permitting no more than a simultaneous change of sign to two of the quantities l, m, n.

§5. *The Quadratic Forms l, m, n considered as Covariants of the Sextic T.*

The forms l, m, n are, as we have stated already, combinants of the pencil $\varkappa f + \lambda h$, or covariants of the sextic $t = T$. We have, according to (31),

$$
T^2 = 4\,G.l^2.m^2.n^2 = \tfrac{4}{3}\,G.\{l^6 + m^6 + n^6\}, \tag{35}
$$

$$
\begin{aligned}
\Phi &= \tfrac{1}{2}\,(T,\ T)_2 = \tfrac{4}{9}\,G.\{m^2n^2 + n^2l^2 + l^2m^2\} = \\
&= -\tfrac{2}{9}\,G.\{l^4 + m^4 + n^4\},
\end{aligned} \tag{36}
$$

$$
\Psi = 2\,(T,\ \Phi)_1 = \tfrac{8}{27}\,G\sqrt{G}\,.\,\{m^2 - n^2\}\,\{n^2 - l^2\}\,\{l^2 - m^2\}, \tag{37}
$$

$$
R = \tfrac{1}{36}\,(T,\ T)_6 = \tfrac{4}{27}\,G. \tag{9}
$$

Consequently the biquadratic forms l^2, m^2, n^2 are the roots of the cubic equation

$$
R\,.\,X^3 - * + \tfrac{1}{3}\Phi.X - \tfrac{1}{27}\,T^2 = 0. \tag{38}
$$

The solution of this equation in terms of radicals is facilitated by the existence of a quadratic resolvent without parameter. On account of the identity (10) we find

$$
\begin{aligned}
&3X = \sqrt[3]{\frac{RT^2 + \Psi\sqrt{-R}}{2}} + \sqrt[3]{\frac{RT^2 - \Psi\sqrt{-R}}{2}}\,, \\
&R.\sqrt[3]{(+)}\sqrt[3]{(-)} = -\Phi,\ \sqrt{-R} = \tfrac{2}{9}\sqrt{-3}.\sqrt{G}.
\end{aligned} \tag{39}
$$

The simplification of the solution of (38) contained in this formula is all that can be reached as long as we operate in the system of the sextic T. But if such a form T is the sextic covariant of a quartic f, which is supposed to be rationally known, then our cubic equation has a cubic resolvent (No. 16) which is free from the parameter $(x_1 : x_2)$.*

The quadratic forms l, m, n, defined by (38) and by the equations in §3, are capable of *eight* systems of values, the relations among the forms l, m, n not being changed when we replace l, m, n by $-l, n, m$. Their ambiguity is reduced to *four* after having disposed of the sign of $\sqrt{G}$ (see No. 31), and so it remains after adjunction of e_λ, e_μ, e_ν.

§6. *Other Irrational Covariants of the Sextic T.*

The formulæ hitherto developed give rise to an unlimited number of irrational covariants of T, combinants of the pencil $\varkappa f + \lambda h$. We shall confine ourselves to a consideration of some of the most important ones.

There are first the two *equianharmonic forms* of the pencil $\varkappa f + \lambda h$, or, as we prefer to say in the present connection, the two quartics that are *apolar* with respect to T† and at the same time equianharmonic. There exists no quadratic equation with coefficients belonging to the system of T, the roots of which would be equianharmonic forms of the said pencil; but there are two different quadratic equations whose roots are the *cubes* of such forms.

We denote again, as we did in the theory of the cubic (p. 171), by ε_1, ε_2, ε_3 any cyclical permutation of the three cube roots of unity ε^0, ε^1, ε^2, and by $\bar{\varepsilon}_1$, $\bar{\varepsilon}_2$, $\bar{\varepsilon}_3$ their conjugate values, arranged in the corresponding order. Then the two quartics

$$\begin{aligned} \mathfrak{S} &= -\sqrt{-R}\,.\{\varepsilon_1 l^2 + \varepsilon_2 m^2 + \varepsilon_3 n^2\}, \\ \mathfrak{T} &= \sqrt{-R}\,.\{\bar{\varepsilon}_1 l^2 + \bar{\varepsilon}_2 m^2 + \bar{\varepsilon}_3 n^2\}, \end{aligned} \tag{40}$$

as well as the two quartics

$$\begin{aligned} \mathfrak{S}' &= \varepsilon_1 l^2 + \varepsilon_2 m^2 + \varepsilon_3 n^2, \\ \mathfrak{T}' &= \bar{\varepsilon}_1 l^2 + \bar{\varepsilon}_2 m^2 + \bar{\varepsilon}_3 n^2 \end{aligned} \tag{40b}$$

represent the equianharmonic forms in question.

---

* The equation (38) has occasionally been mentioned in the author's book, "Methoden zur Theorie der ternären Formen," in order to exemplify the notion of irrational covariant. Unfortunately the equation communicated there has erroneous numerical coefficients.

† The "polars" of f and h with respect to t, viz. the forms $(f, t)_4$ and $(h, t)_4$, vanish, as is well known, identically. Consequently the pencil $\varkappa f + \lambda h$ is made up of the quartics which are "apolar" with respect to t.

After a short calculation we find

$$\begin{gathered}\mathfrak{S}^3 = \frac{\Psi + T^2\sqrt{-R}}{2}, \quad \mathfrak{T}^3 = \frac{\Psi - T^2\sqrt{-R}}{2}, \\ \mathfrak{S}.\mathfrak{T} = \tfrac{3}{2} R.(l^4 + m^4 + n^4) = -\Phi,\end{gathered} \tag{41}$$

$$\begin{gathered}\mathfrak{S}'^3 = \frac{RT^2 - \Psi\sqrt{-R}}{2R^2}, \quad \mathfrak{T}'^3 = \frac{RT^2 + \Psi\sqrt{-R}}{2R^2}, \\ \mathfrak{S}'.\mathfrak{T}' = \tfrac{3}{2}(l^4 + m^4 + n^4) = -\frac{\Phi}{R}.\end{gathered} \tag{41b}$$

Hence the forms $\mathfrak{S}$, $\mathfrak{T}$ *are roots of the sextic equation*

$$Y^6 - \Psi Y^3 - \Phi^3 = 0, \tag{42}$$

whereas $\mathfrak{S}'$, $\mathfrak{T}'$ *are roots of the sextic equation*

$$R^3 Y'^6 - R^2.T^2.Y'^3 - \Phi^3 = 0. \tag{42b}$$

Postponing the further consideration of the equianharmonic forms of the pencil $\varkappa f + \lambda h$, we may investigate now the *three harmonic forms* of the said pencil.

Introducing the notation

$$\begin{aligned} L &= \tfrac{2}{3}\sqrt{G}\,(m + n)(m - n) = 2e_\lambda.h + \left(\frac{g_2}{3} - 2e_\lambda^2\right)f, \\ M &= \tfrac{2}{3}\sqrt{G}\,(n + l)\,(n - l) = 2e_\mu.h + \left(\frac{g_2}{3} - 2e_\mu^2\right)f, \\ N &= \tfrac{2}{3}\sqrt{G}\,(l + m)\,(l - m) = 2e_\nu.h + \left(\frac{g_2}{3} - 2e_\nu^2\right)f, \end{aligned} \tag{43}$$

we have the three forms in question, L, M, N being irrational covariants of T of the degree one.

These forms are the roots of the cubic equation

$$Z^3 - * + 3\Phi Z - \Psi = 0, \tag{44}$$

the solution of which in terms of radicals is contained in the formulæ

$$\begin{gathered} Z = \sqrt[3]{\frac{\Psi + T^2\sqrt{-R}}{2}} + \sqrt[3]{\frac{\Psi - T^2\sqrt{-R}}{2}}, \\ \sqrt[3]{(+)}.\sqrt[3]{(-)} = -\Phi. \end{gathered} \tag{45}$$

The mutual relations of the equations (38), (42), (42b) and (45) are obvious, and need no further explanation; we may add, however, the remark that while (44) gives immediately the decomposition of Ψ into its three conjugate factors, by means of the solution of (38) only the product RT^2 is decomposed.

There exists still an interesting system of *four quadratic forms*, each representing certain two vanishing points of Φ, namely, a pair of vanishing points x, y, connected by the relations

$$(tx)^5(ty) = 0, \quad (ty)^5(tx) = 0, \quad (tx)^4(ty)^2 = 0, \quad (ty)^4(tx)^2 = 0.$$

These forms

$$\begin{aligned} W_0 &= \tfrac{2}{3}\sqrt{G}(l + m + n), \\ W_\lambda &= \tfrac{2}{3}\sqrt{G}(l - m - n), \\ W_\mu &= \tfrac{2}{3}\sqrt{G}(-l + m - n), \\ W_\nu &= \tfrac{2}{3}\sqrt{G}(-l - m + n) \end{aligned} \tag{46}$$

are the roots of the biquadratic equation

$$W^4 + 8RT.\, W + 12R\Phi = 0,$$

which has, according to its origin, (38) and (44) as cubic resolvents.

The sum of these forms is zero as well as the sum of their squares; their Jacobians are proportional to the quadratic forms occurring in (43).*

§7. *The Equianharmonic Forms of the Pencil $\varkappa f + \lambda h$.*

While apparently enough has been said as to the determination of the forms l^2, m^2, n^2, and of the differences $m^2 - n^2$, $n^2 - l^2$, $l^2 - m^2$ as irrational covariants of f, the pair of equianharmonic forms in the pencil $\varkappa f + \lambda h$ deserves further attention.

We have already found two different pairs of irrational covariants proportional to the forms in question; both are roots of equations of the 6th degree and *covariants of T*. Operating, however, not in the system of T but in the system of the quartic f, we obtain a simpler solution of the problem, to find the equianharmonic forms of the pencil $\varkappa f + \lambda h$. Now these forms can be represented in two essentially different ways, by the roots of a *quadratic* equation, which defines them as covariants of f (but not as covariants of T). We have simply to decompose the product $g_2\Phi$ into two factors of the form $\alpha f + \beta\phi$, using the identity (13).

---

*Compare Brill, Math. Annalen, v. 20. Gordan u. Kerschensterner, Binäre Formen (Leipzig, 1887), No. 142.

Let us define these factors $\mathfrak{Y}_+$ and $\mathfrak{Y}_-$ by

$$\begin{aligned} 2\mathfrak{Y}_\pm &= 2g_2.h - (3g_3 \pm \tfrac{4}{3}\sqrt{-3}\sqrt{G})f = \\ &= \pm\tfrac{4}{3}\sqrt{-3}\sqrt{G}.\left\{f \pm \frac{\sqrt{-3}}{2\sqrt{G}}\phi\right\} = \\ &= \sum e_\lambda\{2g_2e_\lambda - 3g_3 \mp \tfrac{4}{3}\sqrt{-3}\sqrt{G}\}.l^2 = \\ &= -\tfrac{4}{3}\sqrt{G}.\sum\{e_\mu - e_\nu \pm \sqrt{-3}.e_\lambda\}l^2. \end{aligned} \tag{48}$$

These forms $\mathfrak{Y}_+$, $\mathfrak{Y}_-$ are the roots of the quadratic equation

$$\mathfrak{Y}^2 - 2\phi.\mathfrak{Y} - 3g_2.\Phi = 0. \tag{49}$$

The forms $\mathfrak{Y}_+$, $\mathfrak{S}$, $\mathfrak{S}'$, and in the same way the forms $\mathfrak{Y}_-$, $\mathfrak{T}$, $\mathfrak{T}'$, have the same vanishing points; their ratios are therefore (irrational) invariants of f:

$$\begin{aligned} \mathfrak{Y}_+ &= -\sqrt{-3}.\sqrt[3]{4\sqrt{G} - 3g_3\sqrt{-3}}.\mathfrak{S}, \\ \mathfrak{Y}_- &= \sqrt{-3}.\sqrt[3]{4\sqrt{G} + 3g_3\sqrt{-3}}.\mathfrak{T}, \end{aligned} \tag{50}$$

where

$$\begin{gathered} \sqrt[3]{4\sqrt{G} - 3g_3\sqrt{-3}} = \\ = \frac{e_\mu - e_\nu + e_\lambda\sqrt{-3}}{\varepsilon_1} = \frac{e_\nu - e_\lambda + e_\mu\sqrt{-3}}{\varepsilon_2} = \frac{e_\lambda - e_\mu + e_\nu\sqrt{-3}}{\varepsilon_3}, \\ \sqrt[3]{4\sqrt{G} + 3g_3\sqrt{-3}} = \\ = \frac{e_\mu - e_\nu - e_\lambda\sqrt{-3}}{\bar{\varepsilon}_1} = \frac{e_\nu - e_\lambda - e_\mu\sqrt{-3}}{\bar{\varepsilon}_2} = \frac{e_\lambda - e_\mu - e_\nu\sqrt{-3}}{\bar{\varepsilon}_3}. \end{gathered} \tag{51}$$

These formulæ contain the expressions of $\mathfrak{S}$, $\mathfrak{T}$, $\mathfrak{S}'$, $\mathfrak{T}'$ in the domain of rationality of the quantities e_λ, e_μ, e_ν, $\sqrt{-3}$.

Substituting finally instead of $\mathfrak{Y}_+$, $\mathfrak{Y}_-$, the quantities

$$\mathfrak{Y}'_+ = -\frac{\mathfrak{Y}_+}{\sqrt{-3}\sqrt{G}}, \quad \mathfrak{Y}'_- = \frac{\mathfrak{Y}_-}{\sqrt{-3}\sqrt{G}}, \tag{48b}$$

we find, instead of (49), the equation

$$G.\mathfrak{Y}'^2 - \tfrac{4}{3}G.f.\mathfrak{Y}' - g_2.\Phi = 0, \tag{49b}$$

which defines another set of equianharmonic forms, irrational covariants of f. The place of the covariant ϕ, occurring in (49), is now taken by f itself.

§8. *Connection between Cubic and Quartic.*

It is well known that by interpreting the parameter $\varkappa:\lambda$ of the pencil $\varkappa f+\lambda h$ as a point in the binary domain, we obtain the simple connection between the theory of a quartic and that of a cubic (Clebsch, B. F., §41 and 44). Now from our theory of irrational covariants we deduce a new and simple analytical expression of this connection.

With a change of notation, we denote a point of the said binary domain by ξ, and, referring to our theory of cubics (I, No. 7–12), write*

$$(\lambda\xi)\cong e_\mu-e_\nu,\quad (\mu\xi)\cong e_\nu-e_\lambda,\quad (\nu\xi)\cong e_\lambda-e_\mu, \tag{52}$$

establishing in this way a *one-to-one correspondence* between the binary domain (ξ) and the binary domain the elements of which are the forms of the pencil $\varkappa f+\lambda h$. It follows immediately that

$$\frac{(\lambda'\xi)}{\sqrt{-3}\sqrt{-r}}\cong e_\lambda,\quad \frac{(\mu'\xi)}{\sqrt{-3}\sqrt{-r}}\cong e_\mu,\quad \frac{(\nu'\xi)}{\sqrt{-3}\sqrt{-r}}\cong e_\nu, \tag{52b}$$

$$\frac{\delta}{r}\cong-\frac{g_2}{4},\quad \frac{p}{r}\cong\sqrt{G},\quad \frac{q}{\sqrt{-3}.r\sqrt{-r}}\cong\frac{3g_3}{4}. \tag{53}$$

By means of these formulæ, to the vanishing points of the forms l^2, m^2, n^2, and to those of the equianharmonic and harmonic forms of the pencil $\varkappa f+\lambda h$, correspond the vanishing points of a binary cubic and of its quadratic and cubic covariant.

This remarkable theorem which is (save the above analytical expression) due to Clebsch, as it seems, can be generalized in two different ways.

Writing s instead of the variable x in the theory of the cubic, and forming the product $(ps)^3.(\xi s)$, *we have in our formulæ an analytical expression for a transformation of the binary domain by which the pencil $\varkappa f+\lambda h$ is transformed into another pencil of binary quartics, namely, into a pencil of quartics which have three linear factors (λs), (μs), (νs) in common, while the last factor (ξs) alone is variable.* Now forming the invariants g_2' and g_3' of the quartic $\frac{1}{r}(ps)^3.(\xi s)$, we find

$$g_2'\cong\frac{3}{16}r.g_2,\quad g_3'\cong-\frac{3\sqrt{-3}}{64}r\sqrt{-r}.g_3. \tag{54}$$

---

* When establishing a principle of transference like the one above, containing a comparison of two theories of different origin, we may use the sign $\cong$ instead of the ordinary sign of equality $=$.

In the following conclusions we assume that the sign $\sqrt{-3}$ has the same meaning in both theories.

Consequently the absolute invariant $\frac{g_2'^3}{g_3'^2}$ *of the product* $(ps)^3.(\xi s)$ *is identical with the absolute invariant* $\frac{g_2^3}{g_3^2}$ *of the quartic* f.

Or, expressing the same result more geometrically:

If we denote the vanishing points of f, *in a properly chosen arrangement, by* r_0, r_λ, r_μ, r_ν, *and let, by means of a linear transformation, the points* r_λ, r_μ, r_ν *coincide with the vanishing points* λ, μ, ν *of the cubic* p, *the point* r_0 *coincides with the point* ξ, *corresponding to our quartic by means of the formulæ* (52).

Later on, when decomposing f into its linear factors, we shall introduce four linear forms (r_0x), $(r_\lambda x)$, $(r_\mu x)$, $(r_\nu x)$, the vanishing points of which represent exactly the arrangement in question.—

Secondly, we may consider more closely the one-to-four correspondence between the points ξ of the binary domain of the cubic p and the points x of the domain of the quartic f, by means of which the point ξ is transformed into the four vanishing points of f, or more generally (considering ξ as variable), into the vanishing points of a form $\varkappa f + \lambda h$.

We may express the transformation in question by means of an equation which contains variables of either domain, (ξ) or (x), and which is obtained by substituting the above values of e_λ, e_μ, e_ν in the expression (29) of f:

$$0 = (\lambda'\xi)\, l^2 + (\mu'\xi)\, m^2 + (\nu'\xi)\, n^2 \cong \sqrt{-3}\,\sqrt{-r}.(ax)^4. \tag{55}$$

Now, let in this way to the points $\xi, \eta, \zeta, \ldots.$ correspond the forms $(ax)^4$, $(bx)^4$, $(cx)^4$ of our pencil. We find, by means of the formulæ (32) and (9):

$$\begin{aligned} (\sqrt{-3}\,\sqrt{-r})^2.(ab)^4 &\cong -24\,(\delta\xi)(\delta\eta), \\ (\sqrt{-3}\,\sqrt{-r})^3.(ab)^2(ac)^2(bc)^2 &\cong 24\,(q\xi)(q\eta)(q\zeta). \end{aligned} \tag{56}$$

This shows that,—

To a group ξ, η *of the polar system of* δ (*viz. to a pair of points satisfying the condition* $(\delta\xi)(\delta\eta) = 0$) *corresponds in our transformation a couple of conjugate forms* $(ax)^4$, $(bx)^4$ *of the pencil* $\varkappa f + \lambda h$. (Such forms as f and ϕ, satisfying the equation $(ab)^4 = 0$.)

To a group ξ, η, ζ *of the polar system of* q *corresponds in our transformation a set of three forms* $(ax)^4$, $(bx)^4$, $(cx)^4$ *of our pencil, bearing the mutual relation*

$$(ab)^2(ac)^2(bc)^2 = 0.$$

We have evidently still a third theorem of this kind, namely:

If the points ξ, η, ζ form a polar-group of p, the biquadratic covariants ϕ, ψ, χ of the corresponding forms $(ax)^4$, $(bx)^4$, $(cx)^4$ are connected by the equation

$$(\phi\psi)^2(\phi\chi)^2(\psi\chi)^2 = 0,$$

(and vice versa. Of course ψ and χ mean, with respect to $(bx)^4$ and $(cx)^4$ what ϕ means with respect to $(ax)^4$.) Supposing that ξ, η, ζ coincide with one another, and paying attention to the identity $-27g_3(\phi) = 32G^2$, we fall back on our former results.

Finally we mention the theorem:

The point η, corresponding to the Hessian h of f, is the linear polar of the point ξ, corresponding to f itself, with respect to the cubic p.

Namely, we obtain the equation

$$(\lambda'\xi)^2.l^2 + (\mu'\xi)^2.m^2 + (\nu'\xi)^2.n^2 = 0,$$

which determines the vanishing points of h, by eliminating η from the equations

$$(p\xi)^2(p\eta) = \frac{r}{3}\cdot\{(\lambda\xi)^2(\lambda\eta) + (\mu\xi)^2(\mu\eta) + (\nu\xi)^2(\nu\eta)\} = 0$$

and

$$(\lambda'\eta).l^2 + (\mu'\eta).m^2 + (\nu'\eta).n^2 = 0.$$

The equation (55) determines the binary quartic corresponding to a given point ξ. The inverse problem, to find the point ξ when the quartic is known, is solved in a similar way by means of the equation

$$0 = (\lambda\xi).l^2 + (\mu\xi).m^2 + (\nu\xi).n^2 \cong -\frac{3}{2\sqrt{G}}\cdot\phi, \tag{55b}$$

which is, in a certain sense, reciprocal to the first one. In order to find the point ξ corresponding to f we have simply to express that the quartic on the right is conjugate to f.

The results derived here from the consideration of the form (55) can also be obtained otherwise without difficulty. But the method is interesting in itself. We have here a very simple, say degenerate, example of an unlimited number of principles of transference, the study of which leads to results of some importance. The central point in these theories is always a form containing variables of two different domains, the variables of which are submitted to independent linear transformations. In our special case it does not seem necessary to establish the complete system of invariants and covariants of the form (55); we may signalize, however, the essential importance of the question.

§9. *Connection between Cubic and Octahedron.*

The one-to-four correspondence between the binary domain (ξ) of the cubic p and the binary domain (x) of the quartic f, the object of the research contained in §8, can be looked at from still a different standpoint, opening a new view of the theory of the octahedron.

We arrive at it by establishing the following theorem:

The one-to-four correspondence between ξ and x defined by the equation

$$(\lambda'\xi).l^2 + (\mu'\xi).m^2 + (\nu'\xi).n^2 = 0 \tag{55}$$

is identical with the one determined by means of the proportion

$$(\lambda\xi):(\mu\xi):(\nu\xi) = l^2 : m^2 : n^2. \tag{57}$$

Namely, comparing the formula (8), p. 172, and (22), p. 180, and paying regard to the circumstance that when the ratios of l^2, m^2, n^2 are known, four different sets of values are still left for the ratios of l, m, n, we see that (57) actually defines a one-to-four correspondence between ξ and x, the point x being perfectly determined by the ratios of l, m, n.

Writing now in (57) for a moment η instead of ξ, and eliminating l^2, m^2, n^2 from (55) and (57), we obtain

$$0 = (\lambda'\xi)(\lambda\eta) + (\mu'\xi)(\mu\eta) + (\nu'\xi)(\nu\eta) = -3(\xi\eta), \tag{58}$$

that is to say, the point η coincides with ξ, and (55) and (57) represent the same transformation.

It follows from this that we are able to put $(\lambda\xi)$, etc., equal to $\rho . l^2$, etc., ρ denoting a factor of proportionality. This factor we assume, for certain reasons of homogeneity and partly of convenience, equal to $3\frac{\sqrt{-R}}{\sqrt{-r}}$. Substituting now these values of $(\lambda\xi)$, etc., in the formulæ of the theory of the cubic, we obtain immediately the following theorem:

The system of rational and irrational covariants of the cubic p is transformed into the system of rational and irrational covariants of the sextic T by means of the *following substitutions, which are consistent with each other, and represent a one-to-four correspondence between the binary domains of* (ξ) *and* (x):

$$\begin{array}{lll} \sqrt{-r}\,.\,p \cong \sqrt{-R}\,.\,T^2 & & , \\ \delta \cong \Phi \quad , & q \cong \Psi & , \end{array} \qquad (59)$$

$$\begin{array}{ll} (\sigma\xi) \cong \mathfrak{S} \quad , & (\tau\xi) \cong \mathfrak{T} \quad , \\ \sqrt{-r}\,.\,(\sigma'\xi) \cong \sqrt{-R}\,.\,\mathfrak{S}', & \sqrt{-r}\,(\tau'\xi) \cong \sqrt{-R}\,.\,\mathfrak{T}', \end{array} \qquad (60)$$

$$\begin{array}{lll} \sqrt{-r}\,.\,(\lambda\xi) \cong 3\sqrt{-R}\,.\,l^2, & (\lambda'\xi) \cong L & , \\ \sqrt{-r}\,.\,(\mu\xi) \cong 3\sqrt{-R}\,.\,m^2, & (\mu'\xi) \cong M & , \\ \sqrt{-r}\,.\,(\nu\xi) \cong 3\sqrt{-R}\,.\,n^2, & (\nu'\xi) \cong N & . \end{array} \qquad (61)^*$$

This beautiful result is not surprising. The theory of the irrational covariants of the cubic is based upon the syzygy

$$rp^2 + 4\delta^3 + q^2 = 0 \qquad \text{(Nr. 2, p. 171)},$$

whereas the theory of the covariants of the octahedron is derived from the syzygy

$$RT^4 + 4\Phi^3 + \Psi^2 = 0 \qquad \text{(Nr. 10, p. 178)}.$$

From the close resemblance between these formulæ follows, that in both theories we had to perform, step by step, the same operations; and so we cannot wonder at the fact that finally the results can be transformed into each other by a simple principle of transference. We have not thought it convenient, however, to choose this remark as our starting point, for by doing so we should have lost the connection with the results developed in §8, which is an essential feature of our theory.

Our principle of transference appears in two different forms, a rational (59) and an irrational one (60) and (61). In (60) and (61) we may exchange l^2, m^2 and n^2 *ad libitum*; so we see that the formulæ (59), without the supplementary formulæ (60) and (61), represent a transformation 6 to 24, which is decomposed into six different transformations by adjunction of the irrational covariants l^2, m^2, n^2.

Finally, we mention that the involutory linear transformation, which exchanges the vanishing points of p with the corresponding vanishing points of q, has its equivalent in an involutory four-to-four transformation of the binary domain (x) which transforms the vanishing points of T into those of Ψ, and exchanges the irrational covariants of T among each other.

---

* The aspect of these formulæ is somewhat simplified by the supposition $\sqrt{-r} \cong \sqrt{-R}$, which is legitimate, since no relation between r and R follows from (59)-(61). But thus we should only disguise the irrationality, and destroy the homogeneity of our formulæ with respect to symbols of p and T.

§10. *The Linear Factors $(r_\kappa x)$ of the Quartic.*

It is easy to determine a set of four linear forms whose vanishing points are identical with those of the quartic f. Namely the squares of the said forms can be expressed linearly in terms of the quadratic forms l, m, n, which are linearly independent; their discriminants vanish; finally, the fourth powers of the linear forms in question are conjugate to f. By these properties the squares of the said forms are defined, save factors not depending on $x\,(x_1 : x_2)$.

We find it convenient to choose these factors in two different manners, adapted to different purposes, and accordingly to introduce two different sets of notations $(\rho_\kappa x)$, $(r_\kappa x)$ for what we may term "linear factors of f."

Thus we write

$$\begin{aligned}
\sqrt[4]{G}.(\rho_0 x)^2 &= -2\,(r_0 x)^2 = \sqrt{e_\mu - e_\nu}\,.\,l + \sqrt{e_\nu - e_\lambda}\,.\,m + \sqrt{e_\lambda - e_\mu}\,.\,n\,,\\
\sqrt[4]{G}.(\rho_\lambda x)^2 &= -2\,(r_\lambda x)^2 = \sqrt{e_\mu - e_\nu}\,.\,l - \sqrt{e_\nu - e_\lambda}\,.\,m - \sqrt{e_\lambda - e_\mu}\,.\,n\,,\\
\sqrt[4]{G}.(\rho_\mu x)^2 &= -2\,(r_\mu x)^2 = -\sqrt{e_\mu - e_\nu}\,.\,l + \sqrt{e_\nu - e_\lambda}\,.\,m - \sqrt{e_\lambda - e_\mu}\,.\,n\,,\\
\sqrt[4]{G}.(\rho_\nu x)^2 &= -2\,(r_\nu x)^2 = -\sqrt{e_\mu - e_\nu}\,.\,l - \sqrt{e_\nu - e_\lambda}\,.\,m + \sqrt{e_\lambda - e_\mu}\,.\,n\,.
\end{aligned} \tag{62}$$

Here the square roots of the quantities $e_\mu - e_\nu$, etc., may be chosen arbitrarily, whereas $\sqrt[4]{G}$ means the product of the said square roots:

$$\sqrt[4]{G} = \sqrt{e_\mu - e_\nu}\,.\sqrt{e_\nu - e_\lambda}\,.\sqrt{e_\lambda - e_\mu}\,. \tag{63}$$

The quantities $(r_\kappa x)$, as defined herewith, are evidently roots of an equation of the degree 32 in the original domain of rationality of f, whereas the quantities $(\rho_\kappa x)$ satisfy an equation of the 8th degree only, which is the lowest degree possible for linear irrational covariants proportional to the linear factors of f.

In order to establish this equation, we investigate the values of the symmetric functions

$$\begin{aligned}
\sum (r_i x)^2 &= 0 \qquad ,\\
\sum (r_i x)^2\,(r_\kappa x)^2 &= \tfrac{3}{4}\,\frac{\Phi}{\sqrt{G}}\,,\\
\sum (r_i x)^2\,(r_\kappa x)^2\,(r_l x)^2 &= \tfrac{1}{2}\,\frac{t}{\sqrt{G}}\,,\\
(r_0 x)(r_\lambda x)(r_\mu x)(r_\nu x) &= \tfrac{1}{4}\,.f
\end{aligned} \tag{64}$$

Of course the last of these equations is, strictly speaking, not a consequence of the equations (62), which furnish only the value of the product $\Pi(r_\kappa x)^2$. But being able to extract the square root, we may, as we have actually done, choose one of the two values of $\Pi(r_\kappa x)$ at random.

From (64) follows:

The squares of the linear forms $(\rho_\kappa x)$ *are the roots of the biquadratic equation*

$$G.y^4 + * + 6\phi.y^2 + 4t.y + f^2 = 0, \tag{65}$$

which defines the forms $(\rho_\kappa x)$ and $(r_\kappa x)$ as irrational covariants of the degrees $-\frac{1}{2}$ and $\frac{1}{4}$ respectively.

The solution of this equation depends, as we have seen, upon the solution of the cubic equation No. 16, which is free from the parameter $(x_1 : x_2)$, and upon two subsequent operations which are independent of each other: First, upon the extraction of the square root of two of the quantities l^2, m^2, n^2 (see No. 31), which contain the parameter, but are combinants of the pencil $\varkappa f + \lambda h$; secondly, upon the extraction of the three square roots $\sqrt{e_\mu - e_\nu}$, etc., which are again free from the parameter.*

The formulæ (62)–(65) contain not only the decomposition of $\frac{1}{4}f$ or $\frac{f}{\sqrt{G}}$ into linear factors by means of invariant processes, but they solve also the corresponding problem for *all the forms of the pencil* $\varkappa f + \lambda h$: l, m, n being combinants, we have in our formulæ simply to replace G by $G_{\kappa\lambda}$ and $e_\mu - e_\nu$ etc. by $(e_\mu - e_\nu)(\varkappa - \lambda e_\lambda)$ etc. In special cases these expressions may of course be simplified; for instance, when the linear factors of T, Φ, Ψ are in question.

The decomposition of ϕ into linear factors depends upon the solution of the equation

$$g_3^2.y^4 - 4G.g_3.f.y^2 + 2G.g_3.t.y + \frac{G}{4}.\phi^2 = 0; \tag{65b}$$

the linear factors are defined by

$$\sqrt{g_3}.(\rho_0'x)^2 = \sqrt{G}.\{\sqrt{e_\lambda}.l + \sqrt{e_\mu}.m + \sqrt{e_\nu}.n\}, \text{ etc.} \tag{62b}$$

We are further enabled to decompose the forms of the 24th degree contained in the pencil (RT^4, Φ^3, Ψ^2) which are most conveniently written in the special shape

$$4\Psi^2 - g_3^2.T^4 = -\tfrac{1}{27}g_2^3.T^4 - 16\Phi^3$$

---

* It would be desirable to derive this result from the direct solution of (65) in terms of radicals.

into their linear factors. We have first to split up such a form into its six conjugate biquadratic factors; the solution of this problem is derived from the formulæ on p. 193 by means of our principle of transference; secondly, we have to apply the formulæ (62)–(65).*

§11. *The Irrational Invariants* $(r_i r_\kappa)$.

Let us now operate in the domain of rationality defined by the quantities

$$\sqrt{e_\mu - e_\nu},\ \sqrt{e_\nu - e_\lambda},\ \sqrt{e_\lambda - e_\mu};\ l,\ m,\ n.$$

Here the covariants $(r_\kappa x)^2$ are fully determined; besides we know the value of the product $(r_0 x)(r_\lambda x)(r_\mu x)(r_\nu x)$. Hence we have altogether *eight* different sets of values of the linear covariants $(r_\kappa x)$.

Passing, by proper changes of sign, from one of these sets to the others, we notice that the simultaneous invariants $(r_i r_\kappa)$ assume only four different sets of values. This leads to an important remark: *The simultaneous invariants* $(r_i r_\kappa)$ *belong to the domain of the quantities* $\sqrt{e_\mu - e_\nu}$, $\sqrt{e_\nu - e_\lambda}$, $\sqrt{e_\lambda - e_\mu}$.

Indeed, let us calculate the simultaneous invariants and covariants of the quadratic forms $(r_\kappa x)^2$, defined by (62).

First, the Jacobians of any two of these forms will be found by means of the formulæ (20):

$$-2(r_0 r_\lambda)(r_0 x)(r_\lambda x) = \sqrt{e_\mu - e_\nu}.\{\sqrt{e_\lambda - e_\mu}.m - \sqrt{e_\nu - e_\lambda}.n\},$$
$$-2(r_\mu r_\nu)(r_\mu x)(r_\nu x) = \sqrt{e_\mu - e_\nu}.\{\sqrt{e_\lambda - e_\mu}.m + \sqrt{e_\nu - e_\lambda}.n\}.$$

Comparing the product of these two expressions with the equation

$$4(r_0 x)(r_\lambda x)(r_\mu x)(r_\nu x) = f \qquad \text{(No. 64)},$$

we have

$$(r_0 r_\lambda).(r_\mu r_\nu) = -(e_\mu - e_\nu),\ \text{etc.} \tag{66a}$$

Moreover we find, by means of (21),

$$(r_0 r_\lambda)^2 = (r_\mu r_\nu)^2 = (e_\mu - e_\nu),\ \text{etc.,} \tag{66b}$$

$$\left.\begin{aligned} (r_\mu r_\nu)(r_\nu r_\lambda)(r_\lambda r_\mu) &= \sqrt[4]{G}, & (r_\lambda r_0)(r_0 r_\mu)(r_\mu r_\lambda) &= \sqrt[4]{G}, \\ (r_\nu r_\lambda)(r_\lambda r_0)(r_0 r_\nu) &= -\sqrt[4]{G}, & (r_0 r_\mu)(r_\mu r_\nu)(r_\nu r_0) &= -\sqrt[4]{G}. \end{aligned}\right\} \tag{66c}$$

---

* Compare Klein's Icosahedron. The standpoint of this work is, however, not quite identical with ours. Klein operates throughout with special systems of coordinates, whereas in our considerations the coordinates remain perfectly general.

By these equations actually four systems of values are left to the simultaneous invariants $(r_i r_\kappa)$, which differ from one another by changes of sign, and correspond exactly to the available changes of sign in the forms $(r_i x)$.

Now we may choose one of the four sets of values at random, assuming for instance

$$\begin{aligned} (r_\mu r_\nu) &= -(r_0 r_\lambda) = \sqrt{e_\mu - e_\nu}, \\ (r_\nu r_\lambda) &= -(r_0 r_\mu) = \sqrt{e_\nu - e_\lambda}, \\ (r_\lambda r_\mu) &= -(r_0 r_\nu) = \sqrt{e_\lambda - e_\mu}. \end{aligned} \tag{67}$$

Hereby, of course, also the ambiguity of the quantities $(r_\kappa x)$ is lessened, only a simultaneous change of sign being left to the whole set.

Thus in the domain of the quantities $l, m, n, \sqrt{e_\mu - e_\nu}, \sqrt{e_\nu - e_\lambda}, \sqrt{e_\lambda - e_\mu}$, *the linear forms* $(r_\kappa x)$ *can be defined as a two-valued set of four quantities.*

This theorem, which seems to have been overlooked hitherto, becomes important when the connection of the theory of the quartic with the theory of *elliptic functions* is in question. The simplicity of the formulæ by means of which we shall express this connection in a subsequent paper, is partly due to the circumstance that we are able to replace the comparatively complicated formulæ (66) by the simpler formulæ (67). Here, as well as in the case of the last equation (64), where we had a first choice among two possibilities, we have made our assumptions so as to make the formulæ expressing the said connection as simple as possible.

From the formulæ (67) we derive the well-known expressions for the *double-ratios* (anharmonic ratios) *of the four points* $r_0, r_\lambda, r_\mu, r_\nu$:

$$\frac{(r_0 r_\nu) \cdot (r_\lambda r_\mu)}{(r_\lambda r_\nu) \cdot (r_0 r_\mu)} = \frac{e_\mu - e_\lambda}{e_\nu - e_\lambda}, \text{ etc.} \tag{68}$$

The quantities e_λ, e_μ, e_ν are already known at the same time with the products $\psi_\lambda = c \cdot (r_0 x)(r_\lambda x)$, $\chi_\lambda = \frac{1}{c} \cdot (r_\mu x)(r_\nu x)$, etc., c denoting an arbitrary parameter:

$$2(\psi_\lambda, \chi_\lambda)_2 = -3e_\lambda.^* \tag{69}$$

§12. *Further Properties of the Linear Forms* $(r_\kappa x)$.

There is a linear identity among any three binary linear forms, the coefficients of which are the simultaneous invariants of these forms. We may write down the peculiar shape the identities among the forms $(r_\kappa x)$ assume on account of the relations (67):

---

*Compare F. Klein, Math. Annalen, v. 27, p. 459, No. 64. The numerical coefficient $\frac{1}{3}$ in Klein's formula is erroneous. The true value is $-\frac{1}{6}$.

$$\begin{aligned} (r_\mu r_\nu).(r_\lambda x) + (r_\nu r_\lambda).(r_\mu x) + (r_\lambda r_\mu).(r_\nu x) &= 0, \\ (r_\mu r_\nu).(r_0 x) + (r_\lambda r_\mu).(r_\mu x) - (r_\nu r_\lambda).(r_\nu x) &= 0, \\ -(r_\lambda r_\mu).(r_\lambda x) + (r_\nu r_\lambda).(r_0 x) + (r_\mu r_\nu).(r_0 x) &= 0, \\ (r_\nu r_\lambda).(r_\lambda x) - (r_\mu r_\nu).(r_\mu x) + (r_\lambda r_\mu).(r_0 x) &= 0. \end{aligned} \tag{70}$$

These formulæ are special cases of the following set of four identities, containing two sets of variables $x\,(x_1 : x_2)$ and $y\,(y_1 : y_2)$:

$$\begin{aligned} (r_0 x)(r_0 y) + (r_\lambda x)(r_\lambda y) + (r_\mu x)(r_\mu y) + (r_\nu x)(r_\nu y) &= 0, \\ (r_0 x)(r_\lambda y) - (r_\lambda x)(r_0 y) + (r_\mu x)(r_\nu y) - (r_\nu x)(r_\mu y) &= 0, \\ (r_0 x)(r_\mu y) - (r_\mu x)(r_\nu y) + (r_\nu x)(r_\lambda y) - (r_\lambda x)(r_\nu y) &= 0, \\ (r_0 x)(r_\nu y) - (r_\nu x)(r_0 y) + (r_\lambda x)(r_\mu y) - (r_\mu x)(r_\lambda y) &= 0. \end{aligned} \tag{71}$$

These remarkable equations are algebraic consequences of the equations (70); the first among them can also be derived from the first equation in (64), while the others immediately follow from the equations (67). (If, in the last formula (64), we should prefer to write $-f$ instead of f, or if we should replace the solution (67) of the equations (66) by another of the four possible solutions, the equations (70) and (71) would, of course, be changed into a different but similar set of equations.)

Worth noticing are further the expressions of the polars $(lx)(ly)$, etc., in terms of the linear forms $(r_\kappa x)(r_\kappa y)$:

$$(lx)(ly) \begin{cases} = -\dfrac{(r_0 x)(r_0 y) + (r_\lambda x)(r_\lambda y)}{(r_\mu r_\nu)} = \dfrac{(r_\mu x)(r_\mu y) + (r_\nu x)(r_\nu y)}{(r_\mu r_\nu)} \\ = \dfrac{(r_0 x)(r_\mu y) + (r_\nu x)(r_\lambda y)}{(r_\mu r_\lambda)} = \dfrac{(r_\mu x)(r_0 y) + (r_\lambda x)(r_\nu y)}{(r_\mu r_\lambda)} \\ = \dfrac{(r_0 x)(r_\nu y) - (r_\mu x)(r_\lambda y)}{(r_\nu r_\lambda)} = \dfrac{(r_\nu x)(r_0 y) - (r_\lambda x)(r_\mu y)}{(r_\nu r_\lambda)}. \end{cases} \tag{72}$$

To these formulæ we may add the following expressions of the products $(mx)(my).(nx)(ny)$ (which we have already recognized as polars (p. 181)), together with the expressions of the products $(xy).(lx)(ly)$, etc.:

$$\begin{aligned} (r_\nu r_\lambda)(r_\lambda r_\mu).(mx)(my).(nx)(ny) &= (r_\mu x)(r_\nu x).(r_\mu y)(r_\nu y) - (r_0 x)(r_\lambda x).(r_0 y)(r_\lambda y), \\ (r_\nu r_\lambda)(r_\lambda r_\mu).(xy).(lx)(ly) &= (r_0 x)(r_\lambda x).(r_\mu y)(r_\nu y) - (r_\mu x)(r_\nu x).(r_0 y)(r_\lambda y), \end{aligned} \tag{73}$$

The formulæ (72) put the well-known fact into evidence that the vanishing points of the form $(lx)^2$ are harmonically separated by the vanishing points r_0, r_λ and r_μ, r_ν of f. The same thing is expressed by the formulæ

$$(lr_0)(lr_\lambda) = 0, \quad (lr_\mu)(lr_\nu) = 0, \tag{74}$$

or by the formulæ

$$\begin{aligned} (lr_0)(lx) &= -(r_\lambda x), \quad (lr_\mu)(lx) = -(r_\nu x), \\ (lr_\lambda)(lx) &= (r_0 x), \quad (lr_\nu)(lx) = (r_\mu x), \end{aligned} \tag{75}$$

$$4.\big((r_0 x)(r_\lambda x),\ (r_\mu x)(r_\nu x)\big)_1 = (r_\nu r_\lambda).(r_\lambda r_\mu).(lx)^2. \tag{76}$$

Finally, among the expressions of covariants of f in terms of the forms $(r_\kappa x)$, the following expression of the covariant ϕ is worth noticing:

$$-\tfrac{3}{2}\frac{\phi}{\sqrt{G}} = (r_0 x)^4 + (r_\lambda x)^4 + (r_\mu x)^4 + (r_\nu x)^4. \tag{77}$$

It shows immediately that ϕ is conjugate to f.

In all these developments, the linear forms $(r_\kappa x)$ are considered as irrational covariants of a given quartic f. But we may just as well choose a set of four given linear forms as starting point. Adapting these forms to the relations (67) we obtain the same theorems, as stated before, arranged in the opposite order.

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